混合整数规划模型的经典分解方法简介：Benders分解

汇 报 人： 莫鹏里 博士

2022年10月21日

CONTENTS

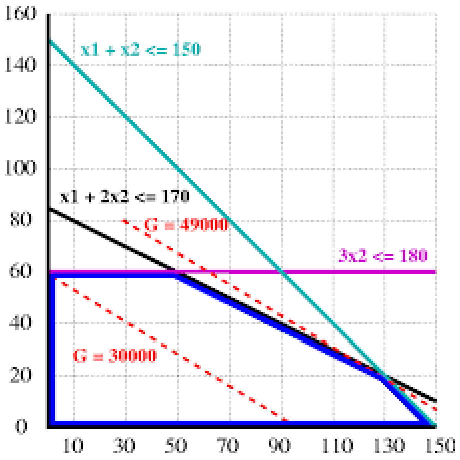
一、交通中的数学优化

◻ 交通中的数学规划问题

■ 交通系统

■ 数学规划

maximize 目标函数 (优化目标)
subject to 约束条件（可行域）



□ 交通中的数学规划问题

■ 运输问题

某个公司从N个产地将物品运往M个销地，每个产地 i 的产量为a_i，每个销地 j 的运入量为b_j。

决策变量

c_{ij} 表示从产地 i 运输到销地 j的成本，$\boxed{x_{ij}}$表示从产地 i 运输到销地 j的产品数量。

$$
\begin{aligned}
&\min \sum_{i=1}^{N}\sum_{j=1}^{M} c_{ij}x_{ij} \\
&s.t. \sum_{j=1}^{M} x_{ij} = a_i, i = 1,2,\dots N \\
&\qquad \sum_{i=1}^{N} x_{ij} = b_j, j = 1,2,\dots M \\
&\qquad x_{ij} \geq 0, i = 1,2,\dots N; j = 1,2,\dots M
\end{aligned}
$$

最小化运输成本　　目标函数

产地约束

销地约束

非负约束

约束条件

□ 交通中的数学规划问题

- **交通场景**
  - 公交线路
  - 公交车
  - 公交乘客

- **交通场景中具有多种优化问题**
  - 公交网络设计问题
  - 公交时刻表排班问题
  - 公交客流分配问题

交通中的数学优化

◻ 交通中的数学规划问题

- **交通场景的规模大**
  - ➢ 公交运营线路有705条
  - ➢ 公交运营车数8395辆
  - ➢ 日均客流560万人次

  （来自南京2016年公交数据）

- **交通模型的数学结构复杂**
  - ➢ **变量复杂**

    连续变量+整数变量-->混合整数
  - ➢ **变量之间的关系复杂**

    线性+非线性

为什么要进行模型分解?

可分解模型的特殊结构

maximize 目标函数项1 + 目标函数项2
subject to 约束条件1(4*5)
约束条件2(5*5)
耦合约束(1*10)

(10*10)=100

maximize 目标函数项1
subject to 约束条件1

(4*5)=20

maximize 目标函数项2
subject to 约束条件2

(5*5)=25

- **如何在确保耦合约束的前提下进行模型分解?**
  - **特定的数学模型结构**

    e.g. 混合整数规划

  - **对于耦合约束信息的交互**

    e.g. Benders分解

二、Benders分解简介

Benders分解简介

□ 什么是混合整数规划?

■ 线性规划

$$\begin{aligned}&\max\ c^T x\\&s.t.\ Ax \le b\\&\qquad x \in R^+\end{aligned}$$

➢ 求解效率高

■ 整数规划

$$\begin{aligned}&\max\ f(x)\\&s.t.\ F(x) \le b\\&\qquad x \in S \subset Z^+\end{aligned}$$

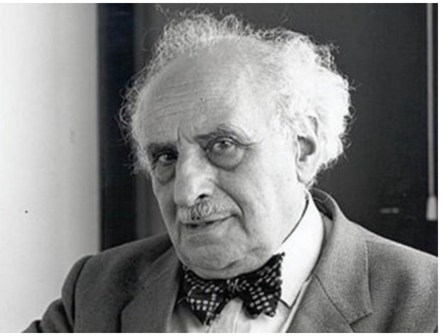
➢ 大多数求解较为困难

■ 混合整数规划

$$\begin{aligned}&\max\ c^T x + f(y)\\&s.t.\ Ax + F(y) \le b\\&\qquad x \in R^+\\&\qquad y \in S\end{aligned}$$

Benders分解简介

□ 经典Benders分解



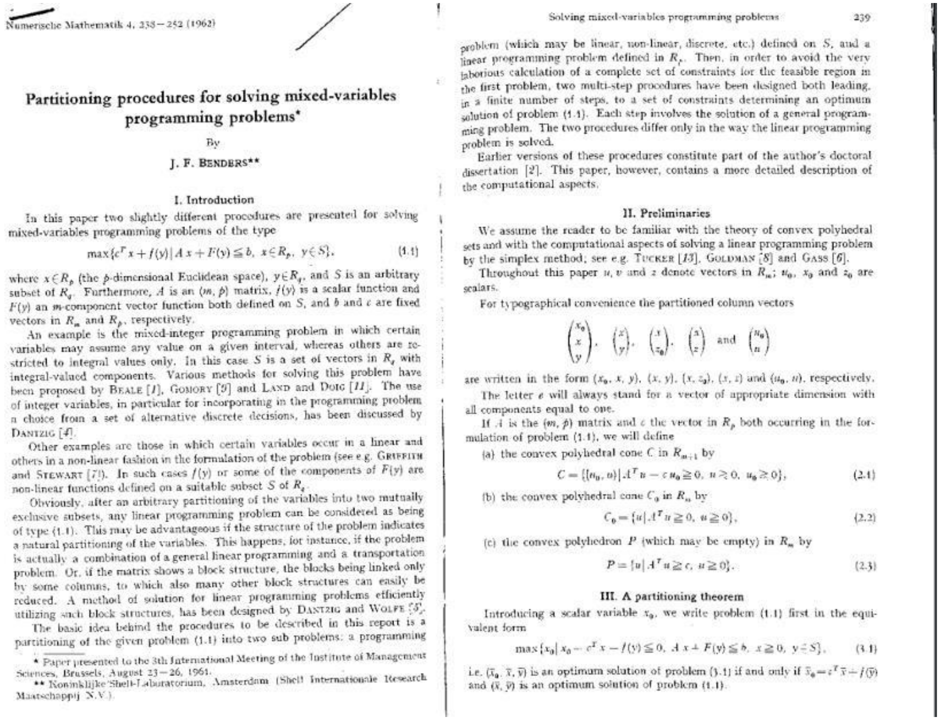
(1924.6.1-2017.1.9)

Jacobus Franciscus (Jacques) Benders

1955, Shell laboratory in Amsterdam

1960, PhD in Utrecht University

1963, Professor of Operations Research at the Eindhoven University of Technology, being the first Professor in the Netherlands in that field



Numerische Mathematik 4, 238–252 (1962)

Partitioning procedures for solving mixed-variables programming problems*

By

J. F. BENDERS**

I. Introduction

In this paper two slightly different procedures are presented for solving mixed-variables programming problems of the type

$$\max\{c^T x + f(y) \mid A x + F(y) \leqq b,\ x \in R_p,\ y \in S\}, \tag{1.1}$$

where $x \in R_p$ (the p-dimensional Euclidean space), $y \in R_q$, and S is an arbitrary subset of R_q. Furthermore, A is an (m, p) matrix, $f(y)$ is a scalar function and $F(y)$ an m-component vector function both defined on S, and b and c are fixed vectors in R_m and R_p, respectively.

An example is the mixed-integer programming problem in which certain variables may assume any value on a given interval, whereas others are restricted to integral values only. In this case S is a set of vectors in R_q with integral-valued components. Various methods for solving this problem have been proposed by BEALE [1], GOMORY [9] and LAND and DOIG [11]. The use of integer variables, in particular for incorporating in the programming problem a choice from a set of alternative discrete decisions, has been discussed by DANTZIG [4].

Other examples are those in which certain variables occur in a linear and others in a non-linear fashion in the formulation of the problem (see e.g. GRIFFITH and STEWART [7]). In such cases $f(y)$ or some of the components of $F(y)$ are non-linear functions defined on a suitable subset S of R_q.

Obviously, after an arbitrary partitioning of the variables into two mutually exclusive subsets, any linear programming problem can be considered as being of type (1.1). This may be advantageous if the structure of the problem indicates a natural partitioning of the variables. This happens, for instance, if the problem is actually a combination of a general linear programming and a transportation problem. Or, if the matrix shows a block structure, the blocks being linked only by some columns, to which also many other block structures can easily be reduced. A method of solution for linear programming problems efficiently utilizing such block structures, has been designed by DANTZIG and WOLFE [5].

The basic idea behind the procedures to be described in this report is a partitioning of the given problem (1.1) into two sub problems: a programming

* Paper presented to the 8th International Meeting of the Institute of Management Sciences, Brussels, August 23–26, 1961.

** Koninklijke Shell-Laboratorium, Amsterdam (Shell Internationale Research Maatschappij N.V.).

Solving mixed-variables programming problems 239

problem (which may be linear, non-linear, discrete, etc.) defined on S, and a linear programming problem defined in R_p. Then, in order to avoid the very laborious calculation of a complete set of constraints for the feasible region in the first problem, two multi-step procedures have been designed both leading, in a finite number of steps, to a set of constraints determining an optimum solution of problem (1.1). Each step involves the solution of a general programming problem. The two procedures differ only in the way the linear programming problem is solved.

Earlier versions of these procedures constitute part of the author's doctoral dissertation [2]. This paper, however, contains a more detailed description of the computational aspects.

II. Preliminaries

We assume the reader to be familiar with the theory of convex polyhedral sets and with the computational aspects of solving a linear programming problem by the simplex method; see e.g. TUCKER [13], GOLDMAN [8] and GASS [6].

Throughout this paper u, v and z denote vectors in R_m; u_0, x_0 and z_0 are scalars.

For typographical convenience the partitioned column vectors

$$\begin{pmatrix} x_0 \\ x \\ y \end{pmatrix},\quad \begin{pmatrix} x \\ y \end{pmatrix},\quad \begin{pmatrix} x \\ z_0 \end{pmatrix},\quad \begin{pmatrix} x \\ z \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} u_0 \\ u \end{pmatrix}$$

are written in the form (x_0, x, y), (x, y), (x, z_0), (x, z) and (u_0, u), respectively.

The letter e will always stand for a vector of appropriate dimension with all components equal to one.

If A is the (m, p) matrix and c the vector in R_p both occurring in the formulation of problem (1.1), we will define

(a) the convex polyhedral cone C in R_{m+1} by

$$C = \{(u_0, u) \mid A^T u - c u_0 \geqq 0,\ u \geqq 0,\ u_0 \geqq 0\}, \tag{2.1}$$

(b) the convex polyhedral cone C_0 in R_m by

$$C_0 = \{u \mid A^T u \geqq 0,\ u \geqq 0\}, \tag{2.2}$$

(c) the convex polyhedron P (which may be empty) in R_m by

$$P = \{u \mid A^T u \geqq c,\ u \geqq 0\}. \tag{2.3}$$

III. A partitioning theorem

Introducing a scalar variable x_0, we write problem (1.1) first in the equivalent form

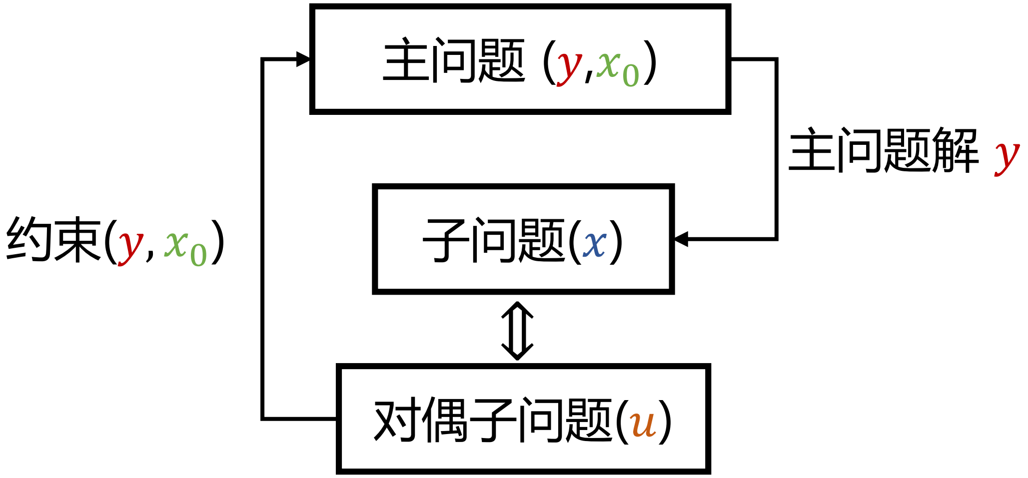
$$\max\{x_0 \mid x_0 - c^T x - f(y) \leqq 0,\ A x + F(y) \leqq b,\ x \geqq 0,\ y \in S\}, \tag{3.1}$$

i.e. $(\bar{x}_0, \bar{x}, \bar{y})$ is an optimum solution of problem (3.1) if and only if $\bar{x}_0 = c^T \bar{x} + f(\bar{y})$ and $(\bar{x}, \bar{y})$ is an optimum solution of problem (1.1).

Benders J F. (1962). Partitioning procedures for solving mixed-variables programming problems. Numerische Mathematik, 4(1), 238-252.

□ 经典Benders分解—— 基本思路

$$
\begin{aligned}
&\max c^T x + f(y) \\
&s.t.\ Ax + F(y) \le b \\
&\qquad x \in R^+ \\
&\qquad y \in S
\end{aligned}
$$



Benders分解简介

◻ 经典Benders分解—— 基本思路

$$\begin{aligned} &\max\ c^T x \\ &s.t.\ Ax \le b \\ &\qquad x \in R^+ \end{aligned} \quad \xleftrightarrow{\text{互为对偶}} \quad \begin{aligned} &\min\ b^T u \\ &s.t.\ A^T u \ge c \\ &\qquad u \in R^+ \end{aligned}$$

原问题（对偶问题）	对偶问题（原问题）
目标函数 max	目标函数 min
变量 $\begin{cases} n\text{个} \\ \ge 0 \\ \le 0 \\ \text{无约束} \end{cases}$	$\left.\begin{matrix} n\text{个} \\ \ge \\ \le \\ = \end{matrix}\right\}$ 约束条件
目标函数中变量的系数	约束条件右边
约束条件 $\begin{cases} m\text{个} \\ \le \\ \ge \\ = \end{cases}$	$\left.\begin{matrix} m\text{个} \\ \ge 0 \\ \le 0 \\ \text{无约束} \end{matrix}\right\}$ 变量
约束条件右边	目标函数中变量的系数

□ 经典Benders分解—— 基本思路

$$\max\ c^T x \quad s.t.\ Ax \le b,\quad x \in R^+$$

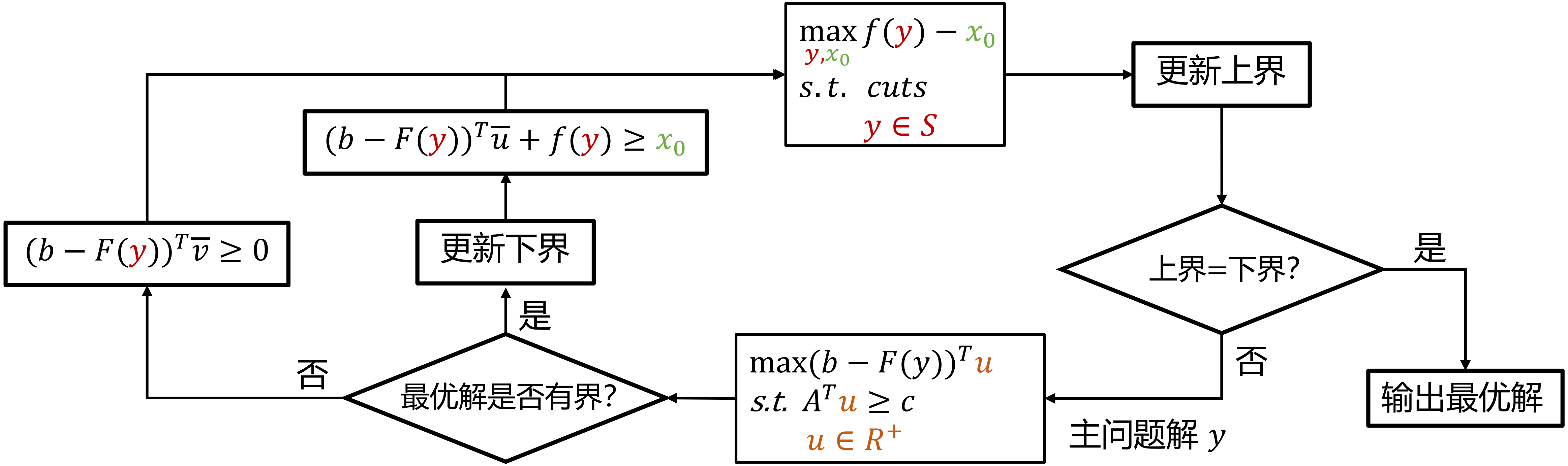
互为对偶 ⟷

$$\min\ b^T u \quad s.t.\ A^T u \ge c,\quad u \in R^+$$

	原问题可行	原问题不可行
对偶问题可行	原问题有最优解 对偶问题也有最优解 且目标函数值相等	**对偶问题目标函数值无界**
对偶问题不可行	原问题目标函数值无界	原问题不可行 对偶问题也不可行

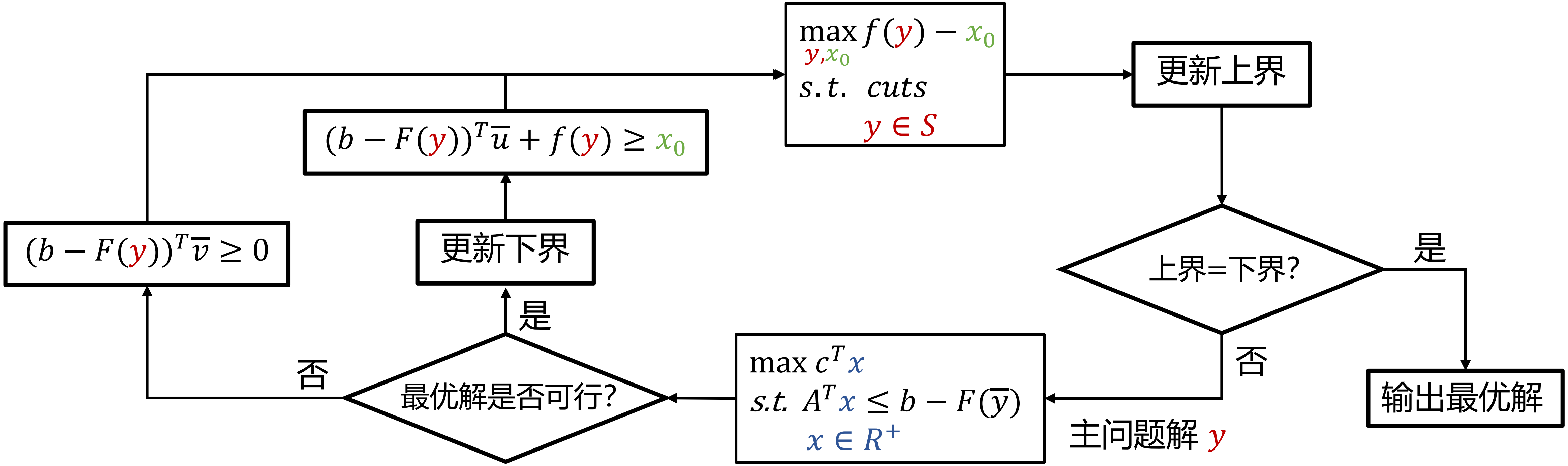
经典Benders分解—— 基本流程

$$\begin{aligned}&\max\ c^T x + f(y)\\ &s.t.\ Ax + F(y) \le b\\ &\qquad x \in R^+\\ &\qquad y \in S\end{aligned}$$



□ 经典Benders分解—— 基本流程

$$
\begin{aligned}
&\max\ c^T x + f(y) \\
&s.t.\ \ Ax + F(y) \le b \\
&\qquad x \in R^+ \\
&\qquad y \in S
\end{aligned}
$$



□ 经典Benders分解—— 代码实战

CPLEX https://www.ibm.com/docs/en/icos/20.1.0?topic=parameters-benders-strategy

GUROBI http://www.gurobi.cn/picexhview.asp?id=90

推荐文章 https://zhuanlan.zhihu.com/p/428706477

https://zhuanlan.zhihu.com/p/572542745

https://mp.weixin.qq.com/s/TSdJJ3bzitmGq1uAB6OSNw

https://mp.weixin.qq.com/s/aRvQKlYIWzhyebYvnvI-Aw

代码技巧 lazy constraints

更新模型参数而无需重写模型

三、Benders分解原理

Benders分解原理

□ 经典Benders分解—— partitioning theorem

原问题

$$\max_{x,y} c^T x + f(y)$$
$$s.t.\ Ax + F(y) \le b$$
$$x \in R^+$$
$$y \in S$$

辅助问题

$$\max_{x_0} x_0$$
$$s.t.\ x_0 - c^T x - f(y) \le 0$$
$$Ax + F(y) \le b$$
$$x \in R^+$$
$$y \in S$$

$(\bar{x}, \bar{y})$是原问题的最优解

$$\overline{x_0} = c^T \bar{x} + f(\bar{y})$$
$$\Updownarrow$$

$(\overline{x_0}, \bar{x}, \bar{y})$是辅助问题的最优解

Benders分解原理

□ 经典Benders分解—— partitioning theorem

原问题

$$\begin{aligned}\max_{x,y}\ & c^T x + f(y)\\ s.t.\ & Ax + F(y) \le b\\ & x \in R^+\\ & y \in S\end{aligned}$$

$$\begin{aligned}\max_{x_0}\ & x_0\\ s.t.\ & x_0 - c^T x - f(y) \le 0\\ & Ax + F(y) \le b\\ & x \in R^+\\ & y \in S\end{aligned}$$

没有 x 的形式

$$\begin{aligned}\max_{x_0}\ & x_0\\ s.t.\ & (x_0, y) \in G\end{aligned}$$

$$G = \bigcap_{(u_0,u)\in C} \{(x_0, y) | u_0 x_0 + u^T F(y) - u_0 f(y) \le u^T b, y \in S\}$$

$$C = \{(u_0, u) | A^T u - c u_0 \ge 0, u \in R^+, u_0 \in R^+\}$$

C 为凸多面体锥，由有限(H)个极射线表示

$$G = \bigcup_{h\le H} \{(x_0, y) | u_0^h x_0 + (u^h)^T F(y) - u_0^h f(y) \le (u^h)^T b, y \in S\}$$

Benders分解原理

□ 经典Benders分解—— partitioning theorem

原问题

$$\max_{x,y} c^T x + f(y)$$
$$s.t.\ Ax + F(y) \le b$$
$$x \in R^+$$
$$y \in S$$

问题3.5

$$\max_{x_0} x_0$$
$$s.t.\ (x_0, y) \in G$$

$$G = \bigcup_{h \le H} \{(x_0, y) | u_0^h x_0 + (u^h)^T F(y) - u_0^h f(y) \le (u^h)^T b, y \in S\}$$
$$(u_0^h, u^h) \in \{(u_0, u) | A^T u - c u_0 \ge 0, u \in R^+, u_0 \in R^+\}$$

(1) 原问题不可行 ⇔ 问题3.5不可行

(2) 原问题可行但没有最优解 ⇔ 问题3.5可行但没有最优解

(3) $(\bar{x}, \bar{y})$是原问题的一个最优解 且 $\overline{x_0} = c^T \bar{x} + f(\bar{y})$ ⇒ $(\overline{x_0}, \bar{y})$是问题3.5的最优解 且 $\bar{x}$是如下问题的最优解

$$\max \{c^T x | A^T x \ge b - F(\bar{y}), x \in R^+\} \quad (问题3.6)$$

(4) 如果$(\overline{x_0}, \bar{y})$是问题3.5的最优解 ⇒ 问题3.6是可行的 且 最优目标值等于$\overline{x_0} - f(\bar{y})$

如果$\bar{x}$是问题3.6的最优解 ⇒ $(\bar{x}, \bar{y})$是原问题的一个最优解 且 最优目标值为$\overline{x_0}$

Benders分解原理

□ 经典Benders分解—— partitioning theorem

(1) 原问题不可行 ⇔ 问题3.5不可行

(2) 原问题可行但没有最优解 ⇔ 问题3.5可行但没有最优解

(3) $(\overline{x}, \overline{y})$是原问题的一个最优解 且 $\overline{x_0} = c^T\overline{x} + f(\overline{y})$ ⇒ $(\overline{x_0}, \overline{y})$是问题3.5的最优解 且 $\overline{x}$是如下问题的最优解

$$\max\{c^T x | A^T x \geq b - F(\overline{y}), x \in R^+\} \quad \text{(问题3.6)}$$

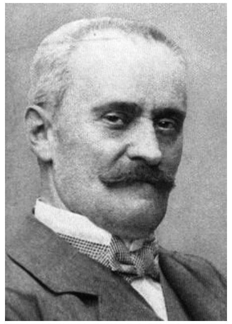
(4) 如果$(\overline{x_0}, \overline{y})$是问题3.5的最优解 ⇒ 问题3.6是可行的 且 最优目标值等于$\overline{x_0} - f(\overline{y})$

如果$\overline{x}$是问题3.6的最优解 ⇒ $(\overline{x}, \overline{y})$是原问题的一个最优解 且 最优目标值为$\overline{x_0}$

$$\boxed{\begin{array}{c} -c^T x \leq -x_0 + f(y) \\ Ax \leq b - F(y) \\ x \in R^+ \end{array}} \text{有可行解} \iff u_0 x_0 + u^T F(y) - u_0 f(y) \leq u^T b \quad \forall (u_0, u) \in C$$

□ 经典Benders分解—— partitioning theorem

$$\boxed{\begin{array}{c} -c^T x \le -x_0 + f(y) \\ Ax \le b - F(y) \\ x \in R^+ \end{array}} \text{有可行解} \quad \Longleftrightarrow \quad u_0 x_0 + u^T F(y) - u_0 f(y) \le u^T b \quad \forall (u_0, u) \in C$$



Julius Farkas

(1847.3.28-1930.12.27)

Farkas' lemma

$$\exists x \in R^+ : Ax = b \qquad or \qquad \exists u \in \{u | A^T u \ge 0\} : b^T u < 0$$

FARKAS, J.,"Die diatonische Dur-Scale wissenschaftlich begründet", Pest, 1870

FARKAS, J., "Die diatonische Dur-Scale wissenschaftlich begründet," J. Reine Angew. Math. 124 (1902), pp. 1-24.

Benders分解原理

□ 经典Benders分解—— partitioning theorem

原问题

$$\max_{x,y} c^T x + f(y)$$
$$s.t.\ Ax + F(y) \le b$$
$$x \in R^+$$
$$y \in S$$

问题3.5

$$\max_{x_0} x_0$$
$$s.t.\ (x_0, y) \in G$$

$$G = \bigcup_{h \le H} \{(x_0, y) | u_0^h x_0 + (u^h)^T F(y) - u_0^h f(y) \le (u^h)^T b, y \in S\}$$

$$(u_0^h, u^h) \in \{(u_0, u) | A^T u - c u_0 \ge 0, u \in R^+, u_0 \in R^+\}$$

(1) 原问题不可行 $\Leftrightarrow$ 问题3.5不可行

(2) 原问题可行但没有最优解 $\Leftrightarrow$ 问题3.5可行但没有最优解

(3) $(\bar{x}, \bar{y})$是原问题的一个最优解 且 $\overline{x_0} = c^T \bar{x} + f(\bar{y}) \Rightarrow (\overline{x_0}, \bar{y})$是问题3.5的最优解 且 $\bar{x}$是如下问题的最优解

$$\max \{c^T x | A^T x \ge b - F(\bar{y}), x \in R^+\} \quad (问题3.6)$$

(4) 如果$(\overline{x_0}, \bar{y})$是问题3.5的最优解 $\Rightarrow$ 问题3.6是可行的 且 最优目标值等于$\overline{x_0} - f(\bar{y})$

如果$\bar{x}$是问题3.6的最优解 $\Rightarrow$ $(\bar{x}, \bar{y})$是原问题的一个最优解 且 最优目标值为$\overline{x_0}$

□ 经典Benders分解—— 计算流程

引理1：

在问题3.5可行 且 y 的可行域有界的前提下：x_0在G上无上界$\Leftrightarrow$ $\boldsymbol{P}=\{u|A^T u \geq c, u \in R^+\}$为空集

$$\text{问题3.5可行} \Rightarrow \text{存在}(x_0^*, y^*) \in G = \bigcap_{(u_0,u)\in C} \{(x_0, y)|u_0 x_0 + u^T F(y) - u_0 f(y) \leq u^T b, y \in S\}$$

$$C = \{(u_0, u)|A^T u - c u_0 \geq 0, u \in R^+, u_0 \in R^+\}$$

$$\text{若}\boldsymbol{P}\text{不是}\emptyset \quad \Rightarrow \quad \begin{array}{c}\text{至少存在}\\ u_0 = \mathbf{1} \\ (1, u) \in C\end{array} \quad \Rightarrow x_0 \leq \max_{y\in Z^+}\{u^T b - u^T F(y) + f(y)\} < \infty$$

$$\text{若}\boldsymbol{P} = \emptyset \quad \Rightarrow \quad u_0 = 0 \quad \Rightarrow \forall x_0, (x_0^*, y^*) \in G \quad \Rightarrow \quad x_0\text{无界}$$

□ 经典Benders分解—— 计算流程

引理2：

如下问题最优解(x_0, y)**是问题3.5最优解**$\Leftrightarrow \min\{(b - F(y))^T u | u \in \boldsymbol{P}\} = x_0 - f(y)$

松弛主问题

$$\begin{array}{l} \max\limits_{x_0} x_0 \\ s.t.\ (x_0, y) \in G\ (Q) \end{array}$$

$$G(Q) = \bigcap_{(u_0,u)\in Q} \{(x_0, y) | u_0 x_0 + u^T F(y) - u_0 f(y) \le u^T b, y \in S\}$$

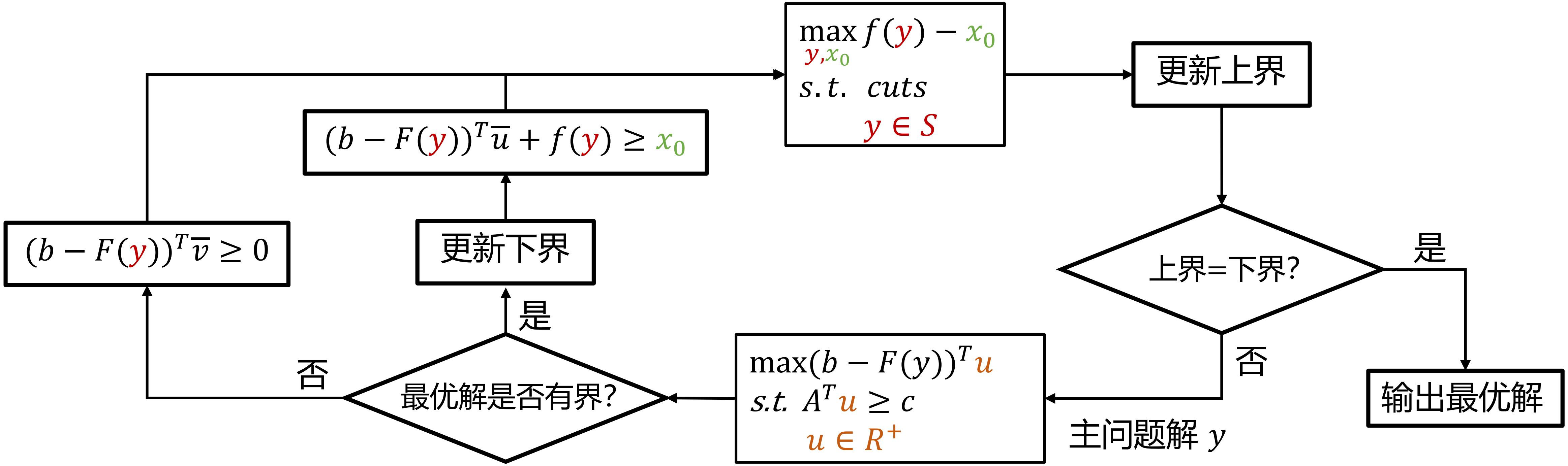
$$Q \subset C = \{(u_0, u) | A^T u - c u_0 \ge 0, u \in R^+, u_0 \in R^+\}$$

$$u_0 x_0 + u^T F(y) - u_0 f(y) \le u^T b \begin{cases} u_0 = 0 \Rightarrow u^T F(y) \le u^T b \Rightarrow (b - F(y))^T u \ge 0 \\ u_0 = \mathbf{1} \Rightarrow x_0 + u^T F(y) - f(y) \le u^T b \Rightarrow x_0 \le u^T b - u^T F(y) + f(y) \end{cases}$$

Benders分解原理

□ 经典Benders分解—— 基本流程

$$
\begin{aligned}
\max\ & c^T x + f(y) \\
s.t.\ & Ax + F(y) \le b \\
& x \in R^+ \\
& y \in S
\end{aligned}
$$



谢谢您的倾听
敬请批评指正

办公地点：东南大学九龙湖校区交通学院大楼1112室

联系方式：mopengli@seu.edu.cn

SOUTHEAST UNIVERSITY